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The chiral 2-sphere

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Abstract. The two-dimensional surface of a sphere can be parametrized by coordinates representing two charged pions acting as Goldstone bosons of a broken SU_2 symmetry. We construct in full concrete detail, and in a general class of coordinate systems, all the relevant structure forming a framework for this low-energy effective theory.

1. Introduction

It is now some 25 years since nonlinear chiral $SU_2 \times SU_2$ Lagrangians were introduced to study the experimental consequences of the emergence of three massless pions as Goldstone bosons, and the results have been clearly exhibited in excellent review articles [1, 2]. Later a very detailed and remarkably successful effective chiral Lagrangian perturbative treatment of low-energy physics was proposed by Gasser and Leutwyler [3, 4] and is now regarded as standard in the field. In such schemes the transformations of the Goldstone bosons are nonlinear and general treatments of the required coset-space mathematics are well established and elegant in form [5, 6]. Also, the consequential construction of invariant nonlinear Lagrangians is standard and well known [7, 8].

From time to time, as in the case of effective chiral Lagrangians mentioned above, there are developments in physics which create a resurgence of interest in the structure. This was particularly the case when supersymmetric σ models were first taken seriously [9, 10] because of similarities of their properties in two dimensions with the structure of four-dimensional gauge theories [11]. The generalization to CP_N models in four dimensions [12] followed swiftly, and a seminal paper by Zumino [13] showed the central place of geometry in the models, with the Kähler metric of complex manifolds providing an elegant description of the supersymmetry. There then followed a decade in which the main focus of attention was on preon like models in which the dominant theme was that the supersymmetry helped to ensure the existence of light fermions by relating them to bosons which were in turn kept light by the Goldstone theorem. A general analysis of the required features can be obtained by working backwards through the literature from the references given in papers by Kotcheff and Shore [14], and by Buchmüller and Lerche [15], both of which are written with authority and also have fine introductory sections.

Recently there have been two developments which suggest a yet further resurgence of interest in these topics. The electric–magnetic duality conjectured by Olive and Montonen [16] several years ago, and shown by Osborn [17] to be related to $N = 4$ supersymmetric gauge theories, has emerged in a generalization in the work of Sen [18].

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Moreover, this seems to play an important role in the work of Seiberg and Witten [19] involving $N = 2$ supersymmetric gauge theory in four dimensions. Crucial to an understanding of this work seems to be the nonlinear Goldstone bosonic parameterization of a manifold which supports a Kähler metric thus allowing the supersymmetric partners of the Goldstone bosons to be introduced and their interactions to be studied. The number of low-energy bosons required by the supersymmetry is larger than the number of true Goldstone bosons resulting from the spontaneous breaking of the underlying non-supersymmetric theory. A similar geometrical structure is then required to treat the sector containing monopoles and their interactions. Although these topics are well understood and have been clearly presented, the technical challenge is still formidable for new workers hoping to enter this exciting and rapidly moving research field. The chiral sphere model presented here is just the general coordinate treatment of the bosonsic manifold underlying quite literally the simplest supersymmetric sigma model which can be constructed. It serves as a useful toy model, retaining many features of much more realistic schemes, which will allow investigation of the salient properties with the minimum of mathematical technique required. On an apparently unrelated front, following the emergence of the supersymmetric standard model as a major candidate for physics beyond the standard model, has come the realization that supersymmetry may appear in nature at energies which may soon be experimentally accessible. Thus a supersymmetric extension of chiral perturbation theory becomes of real interest. Already, two attempts have been made in this direction [20, 21] both based on linear supersymmetric models in which the symmetry is broken (but the supersymmetry preserved) as the Higgs mass becomes infinite. Of course, even if only two flavours of quarks are considered, the simplest bosonic model is based on the coset space formed by the quotient of chiral $SU_2 \times SU_2$ by the central vector SU_2. Unfortunately, this is not a Kähler manifold and has to be extended, by the inclusion of extra coordinates interpreted as fields for pseudo-Goldstone bosons, before spinor partners of the Goldstone bosons can be introduced and a supersymmetric scheme can be described. Not only is this technically difficult (although well described by Itoh *et al* [22] following the prescription by Bando *et al* [23] but the couplings of the superpartners are not unique. Once again the chiral sphere provides a simple concrete model which has unique couplings of superpartners because it is based on a Kähler manifold. Moreover, this time the chiral sphere structure is uniquely embedded in the larger more physical scheme, thus providing not only qualitative understanding but also direct physical contact. We stress again that the sphere is the simplest of the Kähler manifolds.

It seems that supersymmetric sigma models are ripe for further investigation, and obviously the simplest underlying Kähler manifold is the 2-sphere [24]. What is presented in this paper is a direct treatment of the manifold structure, the nonlinear transformation laws of the Goldstone bosons, and the construction of the invariant Lagrangians, all in a general class of coordinate systems. Curiously, although the 2-sphere has been extensively studied this does not seem to have been recorded before. There are, of course, versions in coordinates resulting from constrained linear σ models, treatments in exponential (standard) coordinates, projective coordinate presentations, and most importantly stereographic coordinate representations revealing the Kähler structure. Our general coordinate treatment includes and relates all of these, and we believe it reveals the structure in much the same way that covariant notation clarifies special relativity. We shall show how this model, although not physical, is uniquely embedded in chiral $SU_2 \times SU_2$ (which is indeed of direct physical interest as noted above) and retains many of the relevant features thus allowing them to be studied in a much simpler and concrete way. It is a very useful theoretical laboratory.

2. The chiral sphere

We start this section by reviewing [25] the structure of chiral $SU_2 \times SU_2$ to establish notation. The transformation of the fundamental (quark) multiplet is specified by

$$q \rightarrow q - \mathrm{i}\theta_i \tfrac{1}{2}\tau^i q - \mathrm{i}\phi_i \tfrac{1}{2}\tau^i (\mathrm{i}\gamma_5) q \tag{1}$$

to lowest order in the real parameters θ_i and ϕ_i, $1 \leqslant i \leqslant 3$, where τ^i are the familiar Pauli matrices. Note the extra $(\mathrm{i}\gamma_5)$ factors in the final terms which are included to ensure that the Goldstone bosons of this scheme will be pseudoscalar. We emphasize that this is precisely the usual symmetry of the quark and gluon QCD Lagrangian in the 2 flavour case (ignoring $U(1)$ complications) which leads to the familiar low-energy approximation to hadronic physics [1–4]. Our γ_5 is not Hermitian, but self-barred, so that under the transformations in equation (1) the quark mass term $m\bar{q}q$ is not invariant and so should not appear in the unbroken Lagrangian, whereas the kinetic term proportional to $\bar{q}\gamma^\mu\partial_\mu q$ is invariant because the γ^μ anticommute with the γ_5 in the axial generators. The crucial step in describing the Goldstone bosons is to parametrize the coset space defined by the quotient of the $SU_2 \times SU_2$ by the vector SU_2 parametrized by the θ^i alone. This takes the simple form

$$\hat{L} = \exp\left\{-\tfrac{1}{2}\mathrm{i}\theta n_i \tau^i (\mathrm{i}\gamma_5)\right\} \tag{2}$$

where the Goldstone fields are described by

$$M^i = M n^i \tag{3}$$

with

$$(n^i)(n^i) = 1 \tag{4}$$

so that

$$(M^i)(M^i) = M^2 \tag{5}$$

and θ is an arbitrary dimensionless function of the quotient of M by a constant f_π. Provided that θ is proportional to this quotient in the limit of small fields then f_π is proportional to the pion decay constant. This arbitrariness may be viewed as the freedom to change coordinate systems on the coset space, or to redefine the field variables describing the mesons. Notice that the Goldstone fields M^i really do serve to describe three pseudoscalar pions as usual. This notation is reserved for this general coordinate system (as opposed to π^i for the nonlinear σ-model coordinates, say), and we stress again that if θ is an arbitrary function of M (normalized to M for small fields) then all coordinate systems (with overlapping coordinate patches, i.e. not prohibited by singularities) are incorporated in this one description. (Examples will be given in section 4 of this paper.) If we define projection operators by

$$P_L = \tfrac{1}{2}(1 + \mathrm{i}\gamma_5) \tag{6}$$

and

$$P_R = \tfrac{1}{2}(1 - \mathrm{i}\gamma_5) \tag{7}$$

so that

$$P_L P_L = P_L \tag{8}$$

$$P_R P_R = P_R \tag{9}$$

$$P_L P_R = 0 = P_R P_L \tag{10}$$

and

$$P_L + P_R = 1 \tag{11}$$

then we can rewrite equation (2) as

$$\hat{L} = LP_L + L^{-1}P_R \tag{12}$$

where L is unitary and the γ_5 dependence is now contained solely in the projection operators. It is then clear that we can deal with

$$L = \exp\left\{-\tfrac{1}{2}\mathrm{i}\theta n_i \tau^i\right\} \tag{13}$$

and reinstate the γ_5 factors only when wishing to consider the explicit couplings of the Goldstone bosons to matter fields. The action of a group element g (of $SU_2 \times SU_2$) on the coset space can be specified by [7]

$$gL = L'h \tag{14}$$

where

$$L'(M_i) = L(M'_i) \tag{15}$$

specifies the nonlinear transformations of the Goldstone boson fields,

$$h = \exp\left\{-\tfrac{1}{2}\mathrm{i}\lambda_i \tau^i\right\} \tag{16}$$

and the λ_i depend on the fields and the group parameters. What we have are nonlinear transformations among the M_i (which give a realization of the group) which are linear under the action of the SU_2 subgroup, thus neatly describing a situation where the full group is still realized, but in a manner well suited to spontaneous breaking to the subgroup. The Goldstone bosons are a linear representation of the SU_2 subgroup only. Although the procedure extends to other representations, for our present purposes it will be sufficient to stay mostly in the fundamental representation.

We are now ready to discuss the chiral SU_2 structure embedded in this framework. Consider the subgroup of the chiral $SU_2 \times SU_2$ group specified in equation (1) by retaining only the parameters θ_3 and ϕ_A, with $A = 1$ and 2. Obviously this is an SU_2 subgroup, and we call it chiral SU_2 in recognition of the $(\mathrm{i}\gamma_5)$ factors with the τ^A generators. Clearly the τ^3 generates a U_1 subgroup, so that the coset space obtained by the quotient of chiral SU_2 by this U_1 is parametrized by coordinates M_A, $A = 1$ and 2, which can be viewed as describing two Goldstone pseudoscalars. Notice that the embedding of this SU_2/U_1 structure in the $\frac{SU_2 \times SU_2}{SU_2}$ structure is uniquely specified. Moreover, if we set M_3 and n_3 to zero in our previous discussion, then

$$L = \exp\left\{-\tfrac{1}{2}\mathrm{i}\theta n_A \tau^A\right\} \tag{17}$$

and set $\lambda_A = 0$, so

$$h = \exp\left\{-\tfrac{1}{2}\mathrm{i}\lambda_3 \tau^3\right\} \tag{18}$$

where θ is now an arbitrary function of

$$M^2 = M_1^2 + M_2^2 \tag{19}$$

which when M_3 becomes zero remains as the only independent scalar. Although we realize that many readers will instantly appreciate the nature of this embedding, experience has taught us that confusion often arises at this point and we hope that a more detailed discussion will not divert readers too far from the real theme. Suppose, in the quark model, we define the vector and axial currents, as usual, by

$$V_i^\mu = \bar{q}\gamma^\mu \tfrac{1}{2}\tau_i q \qquad \text{and} \qquad A_i^\mu = \bar{q}\gamma^\mu \tfrac{1}{2}\tau_i(\mathrm{i}\gamma_5)q \tag{20}$$

and implement the transformations in equation (1) by charges

$$Q_i^V = \int V_i^0 \, \mathrm{d}^3\boldsymbol{x} \qquad \text{and} \qquad Q_i^A = \int A_i^0 \, \mathrm{d}^3\boldsymbol{x} \tag{21}$$

by using free field commutation relations. Naturally, while the symmetry is unbroken, the charges are time independent as a result of being constructed from the time components of the conserved Noether's currents. The chiral $SU_2 \times SU_2$ can now be written as

$$[Q_i^V, Q_j^V] = \mathrm{i}\varepsilon_{ijk} Q_k^V \tag{22}$$

which is the algebra of the central (vector) subgroup, together with

$$[Q_i^V, Q_j^A] = \mathrm{i}\varepsilon_{ijk} Q_k^A \tag{23}$$

confirming that the axial charges are in a three-dimensional representation of the vector subalgebra, and

$$[Q_i^A, Q_j^A] = \mathrm{i}\varepsilon_{ijk} Q_k^V \tag{24}$$

showing the closure of the axial parts of the algebra into the vector subalgebra and revealing the symmetric space structure which clarifies the coset space construction we introduced earlier. Now where is the chiral SU_2 algebra embedded? Obviously, as there are only two inequivalent types of SU_2 in $SU_2 \times SU_2$, this one must be equivalent to one of the more obvious ones. Of course the left and right SU_2 algebras defined by the generators

$$Q_i^L = \tfrac{1}{2}(Q_i^V + Q_i^A) \tag{25}$$

and

$$Q_i^R = \tfrac{1}{2}(Q_i^V - Q_i^A) \tag{26}$$

have the property that

$$[Q_i^L, Q_j^R] = 0 \tag{27}$$

so that the centralizer of either of these SU_2 algebras in the $SU_2 \times SU_2$ is the other. This is quite unlike the way in which the vector subgroup is embedded as seen in equations (23) and (24), so that the chiral SU_2 must be equivalent to the vector subgroup. We can take the unitary operator which impliments this equivalence to be

$$U = \exp\left(\tfrac{1}{2}\mathrm{i}\pi P_L \tau^3\right) \tag{28}$$

which induces the mapping

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \rightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{29}$$

which is a trivial relabelling of the form we took with θ_3 and ϕ_A as parameters, and obviously has the correct commutation properties. This equivalence confirms that the coset space identified as the quotient of chiral SU_2 by the U_1 generated by V_3 is indeed the two-dimensional surface of a sphere as we have claimed. Of course, the mapping in equation (29) clearly mixes parity types, so for the physical applications we have in mind the basis of τ_3 with $\tau_A(\mathrm{i}\gamma_5)$ as generators is the appropriate one justifying as it does our notation M_A as two fields describing pseudoscalar mesons, and dictating the form of the couplings to matter fields correspondingly exactly as in the full chiral $SU_2 \times SU_2$ scheme. Note particularly that chiral SU_2 never appears as the denominator of a quotient defining a coset space. It is a subgroup of $SU_2 \times SU_2$ which is equivalent to the vector SU_2 but at no stage is considered as a conserved subgroup in a broken symmetry scenario. Thus the M_i, and subsequently the M_A are always interpreted as fields describing pseudoscalar Goldstone bosons. The point is that chiral SU_2 is a subgroup of chiral $SU_2 \times SU_2$, and when the latter is spontaneously broken to the vector SU_2 (with pseudoscalars M_i) then chiral SU_2 is broken to the U_1 generated by V_3 (with M_A as the pseudoscalars). Describing broken chiral $SU_2 \times SU_2$ supersymmetry is both difficult technically and ambiguous [20–23], but the broken chiral SU_2 scheme (embedded uniquely in all possible broken chiral $SU_2 \times SU_2$ supersymmetrizations) is unambiguously defined in the framework provided by the very simple Kähler structure of the 2-sphere [13].

We can now see the advantages of using this chiral 2-sphere as a model. It is simpler than the chiral $SU_2 \times SU_2$ scheme even in the purely bosonic sector. Moreover, the 2-sphere is a Kähler manifold and so admits a supersymmetric extension in which the Goldstone bosons acquire fermionic (Weyl) partners without yet more quasi-Goldstone bosons and fermions being forced into the model [24]. Also the resulting couplings among the particles are uniquely specified. Contrast this with the situations in [20] and [21] where the number of bosons doubles, as does the number of associated fermions, and finally the couplings involving these new particles are not uniquely specified. Of course, these latter cases are closer to the physics of the real world (they have three pions for example), but the embedded chiral 2-sphere model retains many significant features and is a far more tractable theoretical laboratory. We now present the details of this model.

3. Transformations and invariants

First we establish the transformation laws of the Goldstone fields under chiral SU_2. It is sufficient to work to lowest order in the group parameters and we denote the transformations by

$$g : M_A \to M_A + \theta_3 K_{3A} + \phi_B K_{BA} \tag{30}$$

where K_{3A} and K_{BA} are Killing field components constructed from the M_A themselves. Of course, the action under an element of the U_1 subgroup is linear so that K_{3A} is already known, but we shall let this emerge from our calculations. Expanding equation (14) we see that we need to solve

$$\left[1 - \tfrac{1}{2}\theta_3\tau_3 - \tfrac{1}{2}\phi_B\tau_B\right] L(M) = [L(M) + L_{,A}\theta_3 K_{3A} + L_{,A}\phi_B K_{BA}] \times \left[1 - \tfrac{1}{2}\lambda_3\tau_3\right] \tag{31}$$

where

$$L_{,A} = \frac{\partial L(M)}{\partial M_A} \tag{32}$$

$$\lambda_3 = \theta_3 + \phi_A\lambda_{A3} \tag{33}$$

and we note that in this particular simple example raising and lowering of indices is of no consequence if we preserve the order of indices on the Killing vector fields. It is clear that the calculations require nothing more than the construction of functions of Pauli matrices, but even so a little technique can be helpful. The quantities

$$P^{\pm} = \tfrac{1}{2}(1 \pm n_A\tau_A) \tag{34}$$

share the projection operator properties given in equations (8) to (11) for the P_L and P_R, as can easily be seen because the n_A form a unit vector. This means that equation (17) can be expressed as

$$L = P^{+}\exp\left(-\tfrac{1}{2}\mathrm{i}\theta\right) + P^{-}\exp\left(\tfrac{1}{2}\theta\right) \tag{35}$$

and other functions can be similarly handled. Also, from equation (19) we see that

$$M,_A = n_A \tag{36}$$

and differentiating

$$M_A = M n_A \tag{37}$$

yields

$$M n_{A,B} = \delta_{AB} - n_A n_B \tag{38}$$

so that

$$\begin{aligned} P,^{\pm}_B &= \pm\tfrac{1}{2}\tau_A(\delta_{AB} - n_A n_B) \\ &= \pm\tfrac{1}{2}\left(\tau_B + n_B P^- - n_B P^+\right). \end{aligned} \tag{39}$$

We note that again the tensors $(\delta_{AB} - n_A n_B)$ and $n_A n_B$ have the by now familiar projection operator properties, so that calculations become systematic and straightforward. A little simple algebra applied to equation (31) reveals that

$$K_{BC} = M\cot\theta(\delta_{BC} - n_B n_C) + n_B n_C \frac{\mathrm{d}M}{\mathrm{d}\theta} \tag{40}$$

and

$$K_{3C} = \varepsilon_{3BC} n_B \phi = \varepsilon_{3BC} M_B \tag{41}$$

where ε_{3BC} is the familiar totally antisymmetric Levi–Civita tensor. As noted previously K_{3C} is linear in the M_C, and we recognise the usual rotational transformation of a vector.

We have already found the transformation laws for the Goldstone bosons and, as the reader can easily check, these are identical to those given in [25] when the truncation of variables described in section 2 is applied. Returning to equations (14) and (18) we note, following [5], that if ψ is an irreducible representation of the unbroken subgroup, so that here (keeping to the fundamental representation) we have simply that

$$\psi \to \psi - \tfrac{1}{2}\theta_3\tau_3\psi \tag{42}$$

then under the full group action

$$\psi \to \psi - \tfrac{1}{2}\theta_3\tau_3\psi - \tfrac{1}{2}\mathrm{i}\phi_B\lambda_{B3}\tau_3\psi \tag{43}$$

where

$$\lambda_{B3} = \varepsilon_{BA3} M_A \tan(\theta/2)/M. \tag{44}$$

Note that this transformation law is linear in ψ, but with nonlinear coefficients constructed from M_A; it and its generalizations are known as standard field transformations, and these exhaust all field types. Again the reader can easily check that the result in equation (44) follows trivially from the corresponding result in [25] when our truncation method is applied.

What remains is to show how to construct invariant Lagrangians from the fields we have introduced. It is at this point that objections arise to the direct extraction of further results from [25] by our truncation method. The difficulty is that later results in [25] explicitly use a property that is not available in the chiral SU_2 substructure. In the full chiral $SU_2 \times SU_2$ the Killing vectors can be combined into so called left and right combinations which viewed as matrices $(K^L)_{AB}$ and $(K^R)_{AB}$ are non-singular and can be inverted. Unfortunately, in the chiral SU_2 substructure only K_{AB} and K_{3C} exist so that this trick (which is a useful shortcut) is not directly available. However, as we shall see, K_{AB} itself is non-singular, and by a slight extension of the calculations we do eventually reach the same results.

So what invariants can be constructed? This question was answered elegantly in [7]. The first point is that no invariant can be constructed from the M_A alone. In particular this implies that an invariant mass term is not available for the Goldstone bosons in accordance with the Goldstone theorem. Now consider derivatives of the fields. The key concept is found by rewriting equation (14) in the forms

$$L' = gLh^{-1} \tag{45}$$

and

$$L'^{-1} = hL^{-1}g^{-1} \tag{46}$$

and differentiating the former to obtain

$$\partial_\mu L' = g\left[(\partial_\mu L)h^{-1} + L(\partial_\mu h^{-1})\right] \tag{47}$$

where ∂_μ differentiates the fields M_A with respect to the coordinate x^μ, but g is constant because we are considering only global transformations. From equations (46) and (47) we see

$$\begin{aligned} L^{-1}\left(\partial_\mu L\right) &\to L'^{-1}\left(\partial_\mu L'\right) \\ &= h\left[L^{-1}\left(\partial_\mu L\right)\right]h^{-1} + h\left(\partial_\mu h^{-1}\right) \end{aligned} \tag{48}$$

and recognise that, because h is in the subgroup, the transformation does not mix the coset space and subgroup generators in the algebra. Thus, if we write

$$\begin{aligned} 2\mathrm{i}L^{-1}\left(\partial_\mu L\right) &= \tau_B a_\mu^B + \tau_3 v_\mu^3 \\ &= a_\mu + v_\mu \end{aligned} \tag{49}$$

then equation (48) gives

$$a_\mu \to h a_\mu h^{-1} \tag{50}$$

and

$$\begin{aligned} v_\mu &\to h v_\mu h^{-1} + h\left(\partial_\mu h^{-1}\right) \\ &= v_\mu + h\left(\partial_\mu h^{-1}\right) \end{aligned} \tag{51}$$

where the final simplification in equation (51) follows because the subgroup is Abelian. It follows from equation (50) that the quantity

$$\tfrac{1}{2}\,\mathrm{Tr}\left[a_\mu a^\mu\right]$$

is an invariant, and in fact this is the only invariant which can be made from the Goldstone bosons which involves exactly two derivatives. Usually the notation of a covariant derivative

$$\Delta_\mu M^B = a_\mu^B \tag{52}$$

is introduced and the expression

$$\mathcal{L} = \tfrac{1}{2}(\Delta_\mu M^B)(\Delta^\mu M_B) \tag{53}$$

written for the Lagrangian which has a leading order expansion in fields appropriate for interpretation as a kinetic-energy term. Isham [8] introduced the metric form

$$\mathcal{L} = \tfrac{1}{2}g_{AB}(\partial_\mu M^A)(\partial^\mu M^B) \tag{54}$$

for this Lagrangian, thus giving a geometric understanding in terms of the metric g_{AB} on the coset space manifold. We return briefly to equation (51) to note that if there is a matter field ψ which transforms under the U_1 subgroup so that

$$\psi \to \psi - \tfrac{1}{2}\mathrm{i}\theta_3\tau^3\psi \qquad (55)$$

then [5] shows that under the full action of the chiral SU_2

$$\psi \to \psi - \tfrac{1}{2}\theta_3\tau_3\psi - \tfrac{1}{2}\mathrm{i}\lambda_{A3}\phi_A\tau_3\psi \qquad (56)$$

and so

$$\Delta_\mu\psi = \partial_\mu\psi - \tfrac{1}{2}\mathrm{i}v_\mu^3\tau^3\psi \qquad (57)$$

is a covariant derivative transforming as ψ itself in equation (56), and may be used to form invariant terms involving matter fields in the usual way [5, 7].

In the remainder of this section we derive the expressions for the covariant derivatives and metric by direct manipulation of the Pauli matrices, and remaining strictly within the chiral SU_2 framework. (Of course, we hope to find the results which could be read off from [25] by our truncation scheme.) We start by introducing a little extra calculational device by defining

$$R_{ij} = \tfrac{1}{2}\,\mathrm{Tr}[L^{-1}\tau_i L\tau_j] \qquad (58)$$

where, as before, i and j lie in the range 1–3. Using the same formalism as in equations (31) to (39), we easily establish that

$$R_{AB} = (\delta_{AB} - n_An_B)\cos\theta + n_An_B \qquad (59)$$

$$R_{A3} = \varepsilon_{AB3}n_B\sin\theta = -R_{3A} \qquad (60)$$

and

$$R_{33} = \cos\theta \qquad (61)$$

where the projection operator properties are again noted. From equation (45) we see that the quantities appearing in the covariant derivatives can be expressed as

$$a_{\mu B} = (\partial_\mu M_C)a_{CB} \qquad (62)$$

and

$$v_{\mu 3} = (\partial_\mu M_C)v_{C3} \qquad (63)$$

where

$$a_{CB} = \mathrm{i}\,\mathrm{Tr}[\tau_B L^{-1}L_{,C}] \qquad (64)$$

and

$$v_{C3} = \mathrm{i}\,\mathrm{Tr}[\tau_3 L^{-1}L_{,C}] \qquad (65)$$

which we shall shortly see are particularly convenient forms. Now we return to our defining equation (31) and extract

$$-\tfrac{1}{2}\mathrm{i}\tau_A L = L_{,B}K_{AB} - \tfrac{1}{2}\mathrm{i}\lambda_{A3}L\tau_3 \qquad (66)$$

and we can deduce that

$$R_{AD} = K_{AB}a_{BD} \qquad (67)$$

by premultiplying by $\tau_D L^{-1}$ and taking the trace. Since K_{AB} is non-singular, we can see that

$$a_{FD} = \left(K^{-1}\right)_{FA} R_{AD} \qquad (68)$$

and hence

$$a_{FD} = (\delta_{FD} - n_F n_D)\frac{\sin\theta}{M} + n_F n_D \frac{\mathrm{d}\theta}{\mathrm{d}M} \tag{69}$$

follows from equations (40) and (59). Similarly, returning to equation (66) we can also deduce that

$$R_{A3} = K_{AB} v_{B3} + \lambda_{A3} \tag{70}$$

by premultiplying by $\tau^3 L^{-1}$ and tracing. Hence we find directly that

$$v_{F3} = \frac{2}{M}\sin^2(\theta/2)\varepsilon_{FZ3} n_Z \tag{71}$$

by using equations (44) and (60). Thus, combining equations (57), (63) and (71), we see that

$$\Delta_\mu \psi = \partial_\mu \psi - \frac{\mathrm{i}}{M^2}(\partial_\mu M^C)\varepsilon_{CD3} M^D \sin^2(\theta/2)\tau_3 \psi \tag{72}$$

emerges as the covariant derivative of a standard field ψ, transforming in the same way as ψ itself under the action of the group. Similarly, combining equations (52), (62) and (69), we see that

$$\Delta_\mu M^B = (\partial_\mu M_C)\left\{(\delta^{CB} - n^C n^B)\frac{\sin\theta}{M} + n^C n^B \frac{\mathrm{d}\theta}{\mathrm{d}M}\right\} \tag{73}$$

$$= \frac{(\partial_\mu M_C)}{M^2}\left\{\frac{\sin\theta}{M}(M^2\delta^{CB} - M^B M^C) + \frac{\mathrm{d}\theta}{\mathrm{d}M} M^C M^B\right\} \tag{74}$$

emerges as the covariant derivative of the Goldstone fields M^B. It follows that, from equations (52), (53), (62) and (69),

$$\mathcal{L} = \tfrac{1}{2}(\Delta_\mu M^B)(\Delta^\mu M_B) \tag{75}$$

$$= \frac{1}{2}(\partial_\mu M^C)(\partial^\mu M^D)\left\{\frac{\sin^2\theta}{M^2}(\delta_{CD} - n_C n_D) + n_C n_D \left(\frac{\mathrm{d}\theta}{\mathrm{d}M}\right)^2\right\} \tag{76}$$

$$= \tfrac{1}{2} g_{CD}(\partial_\mu M^C)(\partial^\mu M^D) \tag{77}$$

where the metric is given by

$$g_{CD} = \frac{\sin^2\theta}{M^2}(\delta_{CD} - n_C n_D) + n_C n_D \left(\frac{\mathrm{d}\theta}{\mathrm{d}M}\right)^2 \tag{78}$$

and we note that the lowest order term in the expansion of θ has been taken to be M in order to retain the conventional normalization of the kinetic term of the Goldstone bosons. This completes our task, and we see that all the results can indeed be found from those in [25] by our truncation method. We do realize that we have not given a strict mathematical proof of the relationship between chiral $\frac{SU_2 \times SU_2}{SU_2}$ and the chiral SU_2/U_1 embedded in it. It is, however, gratifying to see that all the results we need do come out directly with exactly the same form as by our truncation method applied to those in [25].

One last footnote. In the ordinary treatment of rotations in 3 dimensions we are all very familiar with the discovery of the transformation laws of spinors of SO_3 resulting from ‘taking the square root’ of the unimodular matrix in SU_2 used to describe the rotation of a 3-vector viewed as a bispinor. Exactly the analogous situation holds when nonlinear realizations are involved rather than linear representations. We would like to stress that what we now present in this respect is in no way new or innovative, it has all been done long ago and with much more generality. But our version is very simple and concrete. It goes back (at least) to Schouten [26] in 1954, and probably well before that date. More recently it has featured in work on chiral Lagrangians with gauged Wess–Zumino terms by Kaymakcalan *et al* [27, 28], and on bosonized Nambu–Jona–Lasinio models by Wakamatsu and Weise [29]. To some extent in these latter papers, but especially in the extensive work of Bando *et al* [30], the resulting nonlinear transformations have been interpreted as revealing a 'hidden' or 'secret' symmetry, in which gauge bosons are postulated to appear dynamically. This appears to have been first attempted by Balachandran *et al* [31]. We would like to stress that our simple effects are precisely as naive as they seem to be, and are in no way intended to be interpreted in this controversial 'secret' or 'hidden' manner. Our work simply describes the spontaneous breaking of global symmetries. If and when it may be used (as suggested earlier) in conjunction with gauge theories it is not expected to be connected with dynamically generated gauge bosons.

Returning to our theme, just as in general relativity where tetrads or vierbeine are introduced to allow the treatment of spinors by 'taking the square root of the metric', here the unitary unimodular square root nature of L versus L^2 can be exploited by introducing Killing vectors for the square root system. This concept is easier to understand in concrete form. From our defining equation (14) we can see that

$$L\, \tilde{g}^{-1} = h^{-1}L' \tag{79}$$

where we have inverted the equation and then applied the involutive outer automorphism $\sim$ which reverses the signs of the generators in the group but not in the subgroup. Multiplying the respective sides of equations (14) and (79) gives

$$gL^2\, \tilde{g}^{-1} = L'^2 \tag{80}$$

in which h has been eliminated thus emphasizing that the action on M_A, specified by K_{BA}, is determined by L^2. In the notation used previously we have

$$\left\{\tau_{A,} L^2\right\} = -2L^2_{,B} K_{AB} \tag{81}$$

as the significant part of the information. We multiply from the left by $L^{-2}\left(K^{-1}\right)_{CA}$ to see that

$$\left(K^{-1}\right)_{CA}\left[L^{-2}\tau_A L^2 + \tau_A\right] = -2\mathrm{i}L^{-2}L^2_{,C} \tag{82}$$

then multiplying from the right by $\frac{1}{2}\tau^B$ and tracing yields

$$\left(K^{-1}\right)_{CA}\left[\delta_{AB} + \tfrac{1}{2}\,\mathrm{Tr}\left(L^{-2}\tau_A L^2\tau_B\right)\right] = -\mathrm{i}\,\mathrm{Tr}\left(L^{-2}L^2_{,C}\tau_B\right) \tag{83}$$

and comparison with equations (58) and (64) makes clear how the square root can be taken. We define

$$\left(k^{-1}\right)_{CA}\left[\delta_{AB} + R_{AB}\right] = \mathrm{i}\,\mathrm{Tr}\left(\tau_B L^{-1}L_{,C}\right) \tag{84}$$

where the sign in taking the square root has been picked for convenience. Then equations (66) and (67) reveal that

$$\left(k^{-1}\right)_{QT} = \left(K^{-1}\right)_{QA} R_{AD}\left([1+R]^{-1}\right)_{DT} \tag{85}$$

which the reader may enjoy confirming, reproduces the obvious inverse of K_{QT} in equation (40) when θ is halved. This clarifies the sense of the square root. In an entirely analogous way we may write

$$\left(k^{-1}\right)_{CB} R_{B3} = \mathrm{i}\,\mathrm{Tr}\left(\tau_3 L^{-1}L'_C\right) \tag{86}$$

and discover

$$\lambda_{A3} = R_{A3} - K_{AB}\left(k^{-1}\right)_{BF} R_{F3}. \tag{87}$$

Substitution of the results from equation (85) and (60) into equation (79) confirms the expression found in equation (71) for v_{F3}, while similar substitutions into equation (87) retrieve the result previously given in equation (44). We therefore confirm the expressions for the transformation properties and covariant derivatives of the standard fields. The results given in this last section are not directly retrievable (as far as we know) by truncation of the results in [25], since the full chiral structure allowed shortcuts to be taken in the calculations in that paper. It is therefore reassuring to find that the results nevertheless still coincide with those resulting from the truncation method, even though in this particular case the justification for such truncation was previously missing.

4. Field redefinitions

We have alluded several times to the generality and utility of our parametrization. Now it is time to see the scheme in action. Consider first the coordinates resulting from a constraint on the singlet in the chiral SU_2 triplet representation. This is directly embedded in the familiar chiral σ-model [32–34], and we shall retain the nomenclature of σ-model coordinates. In chiral SU_2, one introduces a multiplet

$$\Phi = \pi_A \tau^A \gamma_5 + \sigma \tag{88}$$

transforming as a $q\bar{q}$ bispinor, where

$$q \to q - \mathrm{i}\theta_3 \tfrac{1}{2}\tau^3 q - \mathrm{i}\phi_A \tfrac{1}{2}\tau^A (\mathrm{i}\gamma_5) q \tag{89}$$

exactly as in equation (1). It is trivial to see that the group action on this three-dimensional multiplet is

$$\pi_A \to \pi_A + \varepsilon_{A3B}\theta_3\pi_B + \phi_A\sigma \tag{90}$$

$$\sigma \to \sigma - \phi_A\pi_A \tag{91}$$

and we recognise a normal linear representation in which the scalar field σ is a U_1 singlet. (In chiral $SU_2 \times SU_2$ the corresponding multiplet is four dimensional; there are 3 pseudoscalar fields.) In this scheme the nonlinearity results from imposing the chiral SU_2 invariant constraint

$$\sigma^2 + \pi_A\pi_A = f_\pi^2 \tag{92}$$

where f_π is constant, to eliminate the σ field. The transformation law is then

$$\pi_A \to \pi_A + \varepsilon_{A3B}\theta_3\pi_B + \phi_B\left[f_\pi^2 - \pi^2\right]^{1/2}\delta_{AB} \tag{93}$$

where $\pi^2 = \pi_A\pi_A$, and we have arbitrarily selected the positive square root. We emphasize that this is an example of the transformation laws derived earlier for a particular choice of our arbitrary function $\theta(\pi)$. Obviously, the simple form of this transformation law, together with the intuitive feeling for the nonlinearity arising from the constraint, have made this a popular choice in the literature. However, we now turn to the stereographic choice of coordinates used by Zumino [13] to allow the introduction of supersymmetry by emphasizing the Kähler properties of the 2-sphere. Things now look very different. The two real coordinates on the sphere (pseudoscalar fields) are replaced by a single complex variable z. Now the metric is a Hermitian form, and the non zero components are written as

$$g_{z\bar{z}} = \frac{\partial^2 V}{\partial z \partial \bar{z}} \tag{94}$$

where V is a potential function, so that the usual cross derivative constraints [13] are satisfied for this to be a Kähler manifold. In this framework, the nonlinear transformation law for the coordinates takes the form

$$z \to z + \mathrm{i}\theta_3 z + \frac{cw}{2} + \frac{\bar{w}z^2}{2c} \tag{95}$$

where

$$w = \phi_1 + \mathrm{i}\phi_2 \tag{96}$$

c is constant (identifiable as $2f_\pi$), and we note that this transformation law is holomorphic in z. Of course it is simple to change to a more familiar pair of real variables, now x_A $(A = 1, 2)$, by setting

$$z = x_1 + \mathrm{i}x_2 \tag{97}$$

and we find that equation (95) yields

$$x_A \to x_A + \varepsilon_{A3B}\theta_3 x_B + \phi_B \left[\frac{\delta_{AB}(c^2 - x^2)}{2c} + \frac{x_A x_B}{c} \right] \tag{98}$$

where

$$x^2 = x_A x^A \tag{99}$$

and equation (96) has been used. By comparing the δ_{AB} term in equations (30) and (93), using (40), we discover

$$\pi \cot\theta = [f_\pi^2 - \pi^2]^{1/2} \tag{100}$$

and similarly comparing equations (30) and (98), again using (40),

$$2cx \cot\theta = c^2 - x^2 \tag{101}$$

emerges. Direct comparison of these last two results gives

$$\pi^2 = \frac{4c^2 f_\pi^2 x^2}{(c^2 + x^2)^2} \tag{102}$$

or equivalently

$$\pi_A = \frac{2cf_\pi x_A}{c^2 + x^2} \tag{103}$$

as the connection between the two coordinate systems. We note again that c may be identified with $2f_\pi$, when the equality of the two coordinate systems is transparent in the small field limit. This simple example, we hope, makes clear the advantage of working with the general coordinate treatment presented in this paper.

5. Conclusions

We have pointed out that in two active research areas (namely electric–magnetic duality, and the extension of chiral perturbation theory to the supersymmetric domain) the incorporation of supersymmetry forces the inclusion of extra pseudo-Goldstone bosons. The mathematical technique required is daunting for the non-expert, and the resulting effective Lagrangians are not uniquely determined. However the chiral SU_2 model described here is uniquely embedded in the above frameworks and retains many of their more physical features. It is an ideal theoretical laboratory since it concerns the simplest Kähler manifold of all. (The mathematics requires little more than an ability to multiply Pauli matrices.) Finally, as we have shown in section 4, our general coordinate treatment really does yield a very transparent connecting framework incorporating previously unrelated features. We hope the model provides both insight into the general structures involved, and offers an easier route into this research for the uninitiated than is usually the case.

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